ON HOMOLOGICALLY TRIVIAL 3-MANIFOLDS

BY

D. R. McMILLAN, JR.[1]

1. **Introduction.** In [6], R. H. Bing proved that a compact, connected absolute 3-manifold M (i.e., 3-manifold without boundary) is topologically a 3-sphere if each polyhedral simple closed curve in M lies in a topological cube in M. He also raised the following question: Is a compact, connected absolute 3-manifold M a topological 3-sphere if each polyhedral simple closed curve in M can be shrunk to a point in a solid torus (of genus one) in M? This question is answered here in the affirmative by Theorem 2 as an immediate corollary to Theorem 1. The lemmas in the first 4 sections are developed to aid in the proof of Theorem 1. The principal result of this paper, however, is an extension of Theorem 1:

THEOREM 3. *Let M be a compact, connected, absolute 3-manifold such that $H_1(M; Z)=0$ and each polyhedral simple closed curve lies in a regular free-manifold in M, in a homologically trivial manner. Then, M is homeomorphic to S^3.*

A regular *free-manifold* is a punctured cube (see [6]) to whose boundary components have been added orientable handles, at most one to each component (see §3). A simple closed curve J is said to lie *trivially* in a regular free-manifold R if it can be shrunk to a point in R. The simple closed curve J is said to lie in R in a *homologically trivial* manner if J circles each handle of R an even number of times. The group $H_1(M; Z)$ is the 1-dimensional simplicial homology group of M with coefficients in the infinite cyclic group Z. It may be obtained by "abelianizing" the fundamental group of M. To be precise, one should say that J lies in R in a homologically trivial manner (mod 2), but the shorter phrase is more convenient. In all other cases, homological terms will refer to the group Z, unless explicitly stated otherwise. Homology groups will be written additively, and fundamental groups multiplicatively.

Both Theorem 1 of [6] and Theorem 3 of this paper are attempts to characterize S^3 by certain of its algebraic properties. For a brief history of the

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Poincaré Conjecture and related questions, see [6]. Some pertinent references are given at the end of this paper.

The overall plan for the proof of Theorem 1 is the same as for the proof of Theorem 1 of [6]. The reader may find it helpful to consult that paper first. All 3-manifolds considered here will have a fixed triangulation. By [3] and [10], there is no loss of generality in assuming this.

All manifolds mentioned in this paper are to be separable metric, and terms such as "simply-connected," "polyhedral," "piecewise linear," etc., are understood in the sense of [3] and [6], as are the abbreviations "Bd M," "Int M," etc., for a manifold M. A ***punctured disk*** is a 2-cell D minus the sum of the interiors of a finite number of disjoint closed polyhedral 2-cells, each contained in Int D. If X is a topological space, the *cone over* X, $C(X)$, is the space formed by identifying the set $X\times\{1\}$ to a point in the product space $X\times[0, 1]$. One usually identifies $X\times\{0\}$ with X.

2. **Cellular decompositions of a 3-manifold.** Let M be a connected 3-manifold without boundary. A locally-finite collection $T=\{\Delta_j\}$ of polyhedral 3-cells whose union is M constitutes a *proper cellular decomposition of* M if the interiors of distinct Δ_j's are disjoint, the intersection of $4-i$ distinct Δ_j's is a closed i-cell or is empty ($i=0$, 1, 2), and no point of M belongs to more than four Δ_j's. The sum of all points of M which are contained in $4-i$ or more elements of T is called the *i-skeleton of* T, and will be denoted by T_i ($i=0, 1, 2$). Note that T_2 is $\sum$ Bd Δ_j. The following cellular decomposition will have a 1-skeleton that is better suited to the purposes of this paper than is the 1-skeleton of a triangulation.

LEMMA 1. *Every connected absolute* 3-*manifold* M *has a proper cellular decomposition* $T=\{\Delta_j\}$ *such that each point of* T_1 *is of order* 2 *or* 4, $T_1\cdot\Delta_j = T_1\cdot\text{Bd}\,\Delta_j$ *is a nonempty, connected finite subgraph of the polyhedral graph* T_1, *and the boundary of each component of* Bd $\Delta_j-T_1\cdot\Delta_j$ *is a polygonal simple closed curve.*

The decomposition described in the proof of Theorem 2 of [5] for any compact, connected, triangulated 3-manifold without boundary may be used in this case also. In Figure 2 of that paper, the intersection of T_1 with one tetrahedron of a triangulation of M is shown. By [3], every 3-manifold can be triangulated. The decomposition guaranteed by Lemma 1 will be referred to as a *special cellular decomposition of* M.

3. **Some 3-manifolds with boundary in** E^3. Let M be a compact, connected, simply-connected 3-manifold. By Lemma 1, M has a special cellular decomposition T. The class of 3-manifolds with boundary to be described here includes the punctured cubes of [6], yet its members retain the property that if one of them lies in M as a polyhedral subset and contains T_1 in its interior, then some other member has the additional property of containing T_2. It will then be easy to see that M is topologically S^3.

Let E_1 be a polyhedral cube in E^3, and $E_2, \cdots, E_n$, disjoint polyhedral cubes in Int E_1, where $n \geqq 1$. If k is an integer, $1 \leqq k \leqq n$, there are k polyhedral cubes, $C_1, \cdots, C_k$, in E^3 such that C_1 is contained in the closure of the exterior of E_1, $C_i \subseteq E_i$ for $i \leqq k$ and $i \neq 1$, and C_i meets Bd E_i in 2 disjoint polyhedral disks for $1 \leqq i \leqq k$. Then, any space homeomorphic to $R = E_1 - \sum_{i=2}^n \text{Int } E_i + \sum_{i=1}^k C_i$ will be called a *regular free-manifold*. In case $k=0$, R is defined to be $E_1 - \sum_{i=2}^n \text{Int } E_i$. When one wishes to be more specific, R may be completely defined by the ordered pair of integers (k, n), where $0 \leqq k \leqq n$ and $n \geqq 1$. R is said to be obtained by adding orientable handles to a punctured cube. Note that $(1, 1)$ represents a solid torus, and $(0, n)$ represents a punctured cube. In any case, the fundamental group of R is a free group of finite rank k.

The following 3 lemmas are easy consequences of the results found in [2] and [10].

LEMMA 2. *Suppose that R_1 and R_2 are regular free-manifolds, where $R_1 \cdot R_2 =$ Bd $R_1 \cdot$ Bd R_2 is a 2-sphere. Then $R_1 + R_2$ is a regular free-manifold or a 3-sphere.*

LEMMA 3. *Suppose that R is a regular free-manifold, and S is a polyhedral 2-sphere in* Int R. *Then, R is the sum of two polyhedral subsets R_1 and R_2 which are regular free-manifolds, where $R_1 \cdot R_2 =$* Bd $R_1 \cdot$ Bd $R_2 = S$.

LEMMA 4. *Let R be a regular free-manifold and J a polygonal simple closed curve in* Bd R *which separates the component of* Bd R *containing it and which bounds a polyhedral disk D such that* Int $D \subseteq$ Int R. *Then, R is the sum of two polyhedral subsets R_1 and R_2 which are regular free-manifolds, where $R_1 \cdot R_2 =$* Bd $R_1 \cdot$ Bd $R_2 = D$.

The following shows that if a regular free-manifold is cut along a handle, then the result is also a regular free-manifold.

LEMMA 5. *Let R' be a compact, connected 3-manifold with nonempty boundary and let D_1^* and D_2^* be disjoint polyhedral disks in the same component of* Bd R'. *Denote by R the 3-manifold with boundary obtained by identifying D_1^* with D_2^* in an orientable fashion. Then, if R is a regular free-manifold, so is R'.*

Proof. There is a homeomorphism h taking R onto a polyhedral regular free-manifold in E^3, and by [10], h may be supposed to be piecewise linear with respect to some triangulation of R. Let $J_1, J_2, \cdots, J_n$, be polygonal simple closed curves on $h(\text{Bd } R)$ such that J_i separates no component of $h(\text{Bd } R)$, J_i bounds a polyhedral disk D_i, where Int $D_i \subseteq$ Int $h(R)$, no two D_i's intersect, and each D_i is in general position relative to $h(D_1^*) = h(D_2^*)$. It is further required that each component of $h(\text{Bd } R)$ of genus 1, except the one containing $h(\text{Bd } D_1^*) = h(\text{Bd } D_2^*)$, contain exactly 1 of the J_i, and that no J_i lie in the component containing $h(\text{Bd } D_1^*)$.

It will be shown first that the disks D_i can be chosen so that no one of them meets $h(D_1^*)$. Denote by $n(D_i)$ the number of components of $D_i \cdot h(D_1^*)$. Each is a polygonal simple closed curve. If $\sum_{i=1}^n n(D_i)=0$, the desired property holds. Suppose that the D_i have been chosen as above so as to minimize $\sum_{i=1}^n n(D_i)$. If this number is not 0, then for some j there is a polygonal simple closed curve K bounding a disk E in Int $h(D_1^*)$ and a disk F in Int D_j, where Int E misses $\sum_{i=1}^n D_i$. A disk D_j' could then be found so that the collection $D_1, \cdots, D_{j-1}, D_j', D_{j+1}, \cdots, D_n$, satisfies the requirements of the above paragraph, yet $\sum_{i\neq j} n(D_i)+n(D_j') < \sum_{i=1}^n n(D_i)$. The disk D_j' is constructed by adjusting the polyhedral disk $(D_j - \text{Int } F)+E$ in a small neighborhood of E. Hence, $\sum_{i=1}^n n(D_i)=0$.

If $h(R)$ is cut along each of the disks $D_1, \cdots, D_n, h(D_1^*)$, a compact, connected 3-manifold H in E^3 is obtained, each component of whose boundary is a 2-sphere. H is a punctured cube and R' is obtained by adding handles to Bd H. It follows that R' is a regular free-manifold.

If M is a compact, connected 3-manifold with boundary, M is said to be *simply-connected* mod Bd M if $M+C(\text{Bd } M)$ is simply-connected, where $C(\text{Bd } M)$ is the cone over Bd M as defined at the end of §1 (a set X is closed in $M+C(\text{Bd } M)$ if and only if each of the sets $X\cdot M$, $X\cdot C(\text{Bd } M)$ is closed, in M and $C(\text{Bd } M)$, respectively). This is equivalent to the condition that for each polygonal simple closed curve J in Int M, there is a mapping f of a punctured disk D into M such that one component of Bd D maps homeomorphically onto J under f, while every other component of Bd D is mapped by f into Bd M. Note that every 3-manifold with boundary which can be embedded in a simply-connected 3-manifold, and in particular every regular free-manifold, is simply-connected modulo its boundary. The converse is not true. That is, there is an M such that $M+C(\text{Bd } M)$ is simply-connected, but M can be embedded in no simply-connected 3-manifold (see §7).

The above definition, plus a brief discussion of "circling," will be needed for Lemma 6. Let R be a regular free-manifold, and $D_1, \cdots, D_n$, a collection of mutually exclusive polyhedral disks in R each of which meets Bd R in its boundary and such that the 3-manifold obtained by cutting R along these disks is a punctured cube. The D_i form a collection of "handles" for R. If f is a mapping of a simple closed curve J into R, then f can be approximated with any desired accuracy by a piecewise linear homeomorphism h of J into R such that $h(J)\subseteq \text{Int } R$ and $h(J)$ pierces a D_i whenever it meets one. Then, if h is sufficiently close to f, the algebraic linking number of $h(J)$ with Int D_i will be independent of the choice of h and of D_i, and will depend only upon the component of Bd R containing Bd D_i. If this number is k, the mapping f will be said to *circle R k times longitudinally in C_i*, where C_i is the component of Bd R (necessarily of genus 1) containing Bd D_i.

The following shows how one may attach cubes to a regular free-manifold in such a way that the result is a regular free-manifold.

LEMMA 6. *Let R_1 be a regular free-manifold, and R_2 a punctured cube, where $R_1 \cdot R_2$ is an annulus ring A in* Bd $R_1 \cdot$ Bd R_2. *Suppose that $M = R_1 + R_2$ is simply-connected* mod Bd M. *Then, M is a regular free-manifold.*

Proof. Consider the following properties of a simple closed curve J in Bd R_1:

(a) J lies in a simply-connected component of Bd R_1;

(b) J lies in a component of Bd R_1 of genus 1 and bounds a disk in this component;

(c) J lies in a component C of Bd R_1 of genus 1 and circles R_1 once longitudinally in C.

A simple closed curve J with one of these properties is unique in the sense that if J' is another simple closed curve with the same property as J, then there is a homeomorphism of R_1 onto itself taking J onto J'. In the case of property (c), this homeomorphism can be realized by cutting R_1 along one of its handles, twisting one of the resulting "ends" the proper integral number of times, and then rejoining these two ends. It follows that if J is a polygonal simple closed curve in Int A which divides A into 2 annulus rings, and J' is such a simple closed curve in Int A', where A' is also an annulus ring in Bd R_1, and if each of J, J' satisfies the same one of the conditions (a), (b), (c), then there is a homeomorphism of R_1 onto itself taking A onto A'. Hence, it will suffice to show that such a simple closed curve J in Int A has one of the properties (a), (b), or (c).

Suppose first that J circles R_1 n times longitudinally in C, where C is a component of Bd R_1 of genus 1, and $n > 1$. Let $C_1, \cdots, C_k, C$, be the distinct components of Bd R_1 that are of genus 1. There is a collection of mutually exclusive polyhedral 2-spheres $S_1, \cdots, S_k$, in Int R_1 such that for each i, S_i separates C_i from Bd $R_1 - C_i$ in R_1. Now, if H_i is the component of $R_1 - S_i$ that contains C_i, then $T = R_1 - \sum H_i$ is homeomorphic to a solid torus of genus 1 from which has been removed the sum of the interiors of at least k mutually exclusive polyhedral 3-cells, each 3-cell lying in the interior of this torus. Also, $T + R_2$ is simply-connected mod its boundary, since it is contained in $R_1 + R_2$, and each component of Bd$(T + R_2)$ is a 2-sphere. Hence, $T + R_2$ is simply-connected, and J circles T n times longitudinally in C.

Let K be a polygonal simple closed curve in Int T which circles T once longitudinally. Since J bounds a disk in R_2, K can be shrunk to a point in $(T + R_2) - J$. Hence, there is a map f of a punctured disk D into $T - J$ such that the "outer" component of Bd D maps onto K homeomorphically under f, while the "inner" components of Bd D are mapped by f into $C - J$. If F is one of these components, then $f|F$ is a mapping of a simple closed curve into T which circles T a number of times longitudinally in C which is a multiple of n. It follows that the number of times that K circles T longitudinally is a multiple of n, where $n > 1$, a contradiction. The reason for this last statement is that the mapping f induces a homomorphism f^* from the group

$H_1(D; Z)$ into the group $H_1(T; Z)$ (see §1), where these groups are, respectively, the free abelian groups on m generators and on 1 generator. The elements $z_1, z_2, \cdots, z_m$, of $H_1(D; Z)$ corresponding to the oriented "inner" components of Bd D generate $H_1(D; Z)$, and the element z of $H_1(T; Z)$ corresponding to the oriented simple closed curve K generates $H_1(T; Z)$. Since, for $1 \leqq j \leqq m$, $f^*(z_j)$ is some multiple of nz, and since f^* is a homomorphism, $z^{\pm 1} = f^*(z_1 + z_2 + \cdots + z_m)$ is some multiple of nz. This is equivalent to the last assertion about "circling." Hence, J can circle R_1 not more than once in C.

Lastly, suppose that J does not circle R_1 longitudinally in C, yet fails to bound a disk on C, where C is a component of Bd R_1 of genus 1. Then, J bounds a polyhedral disk E such that Int $E \subseteq$ Int R_1 and a polyhedral disk E' such that Int $E' \subseteq$ Int R_2. Now, $E + E'$ is a polyhedral 2-sphere S in Int M, and there is a polygonal simple closed curve L in Int R_1 which meets S in a single point and pierces it there. L would then be a simple closed curve that could not be shrunk to a point in $M + C(\text{Bd } M)$, since the algebraic linking number of L and S is $+1$ or -1. This proves Lemma 6.

4. **Adjusting 3-manifolds in M.** The proof of Theorem 1 follows the pattern of the proof of Theorem 1 of [6]. The reader should compare Lemma 7 of this paper with Lemma 4 of that paper, noting particularly that the proof of the present lemma requires the assumption that M be simply-connected, while that of Lemma 4 of [6] does not.

LEMMA 7. *Let M be a compact, connected, simply-connected 3-manifold with a special cellular decomposition $T = \{\Delta_j\}$. Suppose that M contains a polyhedral subset R which is a regular free-manifold, and that* Int R *contains T_1. Then, M is homeomorphic to S^3.*

Proof. Note that T_1 is a connected, polyhedral finite graph. It may be assumed, since $T_1 \subseteq$ Int R, that Bd R is in general position with respect to each component of $T_2 - T_1$, so that each component of $T_2 \cdot$ Bd R is a polygonal simple closed curve. The proof is by induction on n, the number of components of $T_2 \cdot$ Bd R.

If $n = 0$, then $T_2 \subseteq$ Int R. Then, according to Lemma 3, $R - R \cdot$ Int Δ_1 is a regular free-manifold, and by Lemma 2, $R_1 = (R - R \cdot \text{Int } \Delta_1) + \Delta_1$ is a regular free-manifold whose interior contains Δ_1 or a 3-sphere (the latter in case Bd $R \subseteq$ Int Δ_1). Continuing in this manner, there is R_2, a regular free-manifold or a 3-sphere, such that Int $R_2 \supseteq \Delta_1 + \Delta_2$. Finally, after a finite number of steps, there is a 3-sphere which completely fills M. This completes the proof in case $n = 0$.

Now suppose that $T_2 \cdot$ Bd R consists of n polygonal simple closed curves, where $n \geqq 1$. It suffices to find R', a polyhedral regular free-manifold, such that $T_1 \subseteq$ Int R' and $T_2 \cdot$ Bd R' consists of fewer than n polygonal simple closed curves. Let J be a curve in $T_2 \cdot$ Bd R which bounds a disk D, $D \subseteq T_2 - T_1$, such that Int $D \cdot$ Bd $R=\emptyset$. There are now two cases to consider: $R\cdot$Int $D=\emptyset$ or Int $D \subseteq$ Int R.

Assume now that $R\cdot$Int $D=\emptyset$. There is a polyhedral cube H in M such that Int $D\subseteq$Int H, $H\cdot T_2=D$, and $H\cdot R=$Bd $H\cdot$Bd R is an annulus ring A such that $J\subseteq$Int A. H is obtained by thickening D slightly. Now, $R+H$ is a polyhedral 3-manifold with boundary in M whose interior contains T_1 and which is simply-connected mod its boundary, since M is simply-connected. By Lemma 6, $R+H$ is a regular free-manifold, and clearly $T_2\cdot$Bd$(R+H)$ has one less component than does $T_2\cdot$Bd R.

Suppose next that Int $D\subseteq$Int R. If $J=$Bd D separates the component of Bd R containing it, then Lemma 4 states that $R=R_1+R_2$, where R_1, R_2 are regular free-manifolds such that $R_1\cdot R_2=D$. Now, either $T_1\subseteq$Int R_1 or $T_1\subseteq$Int R_2, say $T_1\subseteq$Int R_1. There is a piecewise linear homeomorphism, h, of M onto M which is fixed except near D, and which pushes Bd R_1 slightly to one side of D in such a manner that $T_1\subseteq$Int $h(R_1)$ and $T_2\cdot$Bd$(h(R_1))$ has fewer components than does $T_2\cdot$Bd R.

Finally, it might happen that Int $D\subseteq$Int R, but $J=$Bd D does not separate the component of Bd R containing it. Let H be a polyhedral cube in R such that $H\cdot T_2=D$, Int $D\subseteq$Int H, and (Bd H)$\cdot$(Bd R) is an annulus ring A containing J in its interior. By Lemma 5, the closure of $R-H$ is the required regular free-manifold R'. This completes Lemma 7.

The following will permit a relaxation of certain polyhedral assumptions in Theorems 1, 2, and 3. Theorem 1 is proved in §5, immediately after Lemma 8. If P is a topological polyhedron in a 3-manifold M, then P is *tame* if there is a homeomorphism of M onto M taking P onto a polyhedral subset of M. If P is a topological 2-manifold which separates M and U is a component of $M-P$, then P is *tame from the U side* if $\overline{U}$ is a 3-manifold with boundary.

LEMMA 8. *Let J be a polyhedral simple closed curve in the interior of a 3-manifold M, and suppose that there is a regular free-manifold R (not necessarily tame) in* Int M *which contains J in such a way that at least one of the following holds*:

(a) *J lies trivially in R*;

(b) *J lies in R in a homologically trivial manner.*

Then, there is a polyhedral regular free-manifold R' containing J in its interior so that J lies in R' in the same manner as it did in R, with respect to the properties (a) *and* (b).

Proof.

1. Exactly as in [6], one may suppose that Bd R is locally polyhedral mod $J\cdot$Bd R (i.e., except possibly at these points). For, by [10], there is a homeomorphism h of R onto a polyhedron in E^3 such that h is locally piecewise linear at each point of Int R. One may then shrink h(Bd R) slightly into

$h(\text{Int } R)$ at each point of $h(\text{Bd } R-J)$, moving no point of $h(J\cdot\text{Bd } R)$, to obtain a regular free-manifold R^* in E^3 such that $h(J)\subseteq R^*\subseteq h(R)$, $\text{Bd } R^* \cdot h(\text{Bd } R)=h(J\cdot\text{Bd } R)=h(J)\cdot\text{Bd } R^*$, and $\text{Bd } R^*$ is locally polyhedral mod $h(J\cdot\text{Bd } R)$. Then, $h^{-1}(R^*)$ will have the desired properties.

2. Suppose now that J meets Int R. There is a polyhedral solid torus T of genus 1 such that $J\subseteq\text{Int } T$, J circles T once longitudinally, and Bd T and Bd R are in general position. One may regard T as a closed ϵ-neighborhood of J, for sufficiently small ϵ. Since there are uncountably many such ϵ's from which to choose, and since Bd T and Bd R will fail to be in general position for only countably many of these, the last requirement is permissible. Further, since $J\cdot\text{Int } R\neq\emptyset$, one can choose ϵ so small that some handle of T lies in Int R. Hence, each of the components of $\text{Bd } T\cdot\text{Bd } R$ (these are polygonal simple closed curves) circles T longitudinally no times. Also, it may be supposed that each of these curves bounds a disk on Bd R.

3. No component of $\text{Bd } T\cdot\text{Bd } R$ circles T meridinally (once) in Bd T. For, suppose this were the case for some component K, and let J^* be a polygonal simple closed curve in Int R which is close to J in the sense that there is a homeomorphism of J^* onto J which moves no point of J^* more than some small positive number δ. One can choose δ so small that J^* circles each handle of R the same number of times as does J, and so that J^* lies in Int T and circles T once longitudinally. Now, J^* bounds a compact, polyhedral 2-manifold D such that $D\subseteq\text{Int } R$. Since $K\subseteq\text{Bd } R$, D misses K. Also, a slight adjustment of T suffices to bring Bd T and Int D into general position, while preserving the other positional properties of Bd T. The simple closed curve K is deformed slightly by this adjustment onto another polygonal simple closed curve, which also circles T once meridianally and lies in $\text{Bd } T\cdot\text{Bd } R$. Thus, it may be assumed that no such adjustment of T is necessary. Now, each component of $\text{Bd } T\cdot\text{Int } D$ is a polygonal simple closed curve which does not circle T longitudinally, and so bounds a disk in T. Hence, J^* bounds a 2-complex in T, which is impossible.

Note that each component of $\text{Bd } T\cdot\text{Bd } R$ bounds a disk in Bd T and a disk in Bd R.

4. A slice can be removed from T along a handle of T contained in Int R, so as to obtain a polyhedral cube H such that $J\cdot\text{Bd } H$ is 2 points a and b, an arc from a to b in J contains $J\cdot\text{Bd } R$, $\text{Bd } H\cdot\text{Bd } R=\text{Bd } T\cdot\text{Bd } R$, and no simple closed curve in $\text{Bd } H\cdot\text{Bd } R$ separates a from b in Bd H. If there is more than 1 simple closed curve in this last set, the excess ones may be removed by a sequence of operations, each operation consisting of attaching a cube to R or cutting R in half (see the proof of Lemma 7). Note that neither of these destroys properties (a) or (b). If $\text{Bd } H\cdot\text{Bd } R$ has only one component, then $R+H$ is the required R'. This completes the proof in case $J\cdot\text{Int } R\neq\emptyset$.

5. Suppose now that $J\subseteq\text{Bd } R$, where Bd R is locally polyhedral mod J, and J is polyhedral. It will be shown in step 5 that Bd R is, in fact, tame.

There are annulus rings A_i ($i=1, 2$) in Bd R such that $A_1 \cdot A_2 = \text{Bd } A_1 \cdot \text{Bd } A_2 = J$ and A_i is locally polyhedral mod J. By Lemma 2.1 of [11], A_i is tame. By Lemma 5.2 of [11], $A_1 + A_2$ is tame, so that Bd R is locally tame at each of its points and, by Theorem 8.1 of [11], is tame. Although the 2 lemmas of [11] used here are stated for the case where M is E^3, they hold also for the more general situation.

The truth of Lemma 8 is now evident.

QUESTION. It seems likely that the 3-manifold R obtained in step 1 of the proof of Lemma 8 is tame. If so, some of the rather artificial restrictions of that lemma could be dropped. In particular, suppose that S is a 2-sphere in E^3, U is a component of $E^3 - S$, and S is tame from the U side. If G is a polyhedral finite graph in $\overline{U}$ such that S is locally polyhedral mod $G \cdot S$, is S tame?

5. **Proof of Theorem 1.**

THEOREM 1. *Let M be a compact, connected 3-manifold such that each polyhedral simple closed curve can be shrunk to a point in a regular free-manifold in M. Then, M is homeomorphic to S^3.*

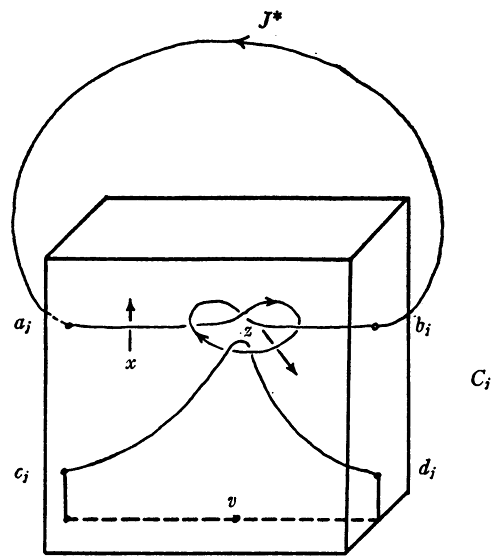

FIG. 1(a)

Proof. According to Lemma 1, M has a special cellular decomposition $T = \{\Delta_j\}$, so that each point of the connected, polyhedral finite graph T_1 is of order two or four. The proof is broken into nine steps.

1. One proceeds exactly as in [6] to approximate T_1 with a polyhedral simple closed curve J. That is, if p_j is a point of T_1 or order 4, there is a polyhedral 3-cell C_j containing p_j in its interior, meeting T_1 in 4 points a_j, b_j, c_j, d_j,

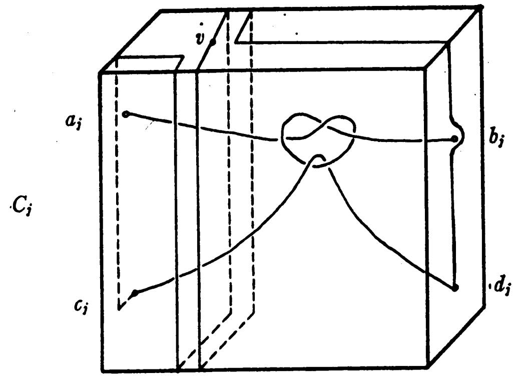

FIG. 1(b)

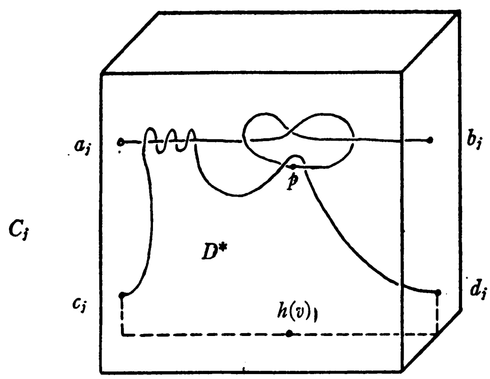

FIG. 1(c)

and such that no 2 C_j's intersect. The required simple closed curve J contains $T_1 - \sum \text{Int } C_i$, is contained in $T_1 + \sum \text{Int } C_i$ and meets C_j in arcs a_jb_j and c_jd_j (see Figures 1(a), 1(b), and 1(c)). The arc a_jb_j is knotted here exactly as in [6], while the arc c_jd_j is unknotted, and the two are linked in such a way that c_jd_j *is not* homotopic in $C_j - a_jb_j$ to any arc in Bd C_j. (See Lemma 6 of [6].) Note, however, that a_jb_j *is* homotopic in $C_j - c_jd_j$ to an arc in Bd C_j.

2. Let R be a polyhedral regular free-manifold containing J in its interior, such that J can be shrunk to a point in R, and such that Bd R and $\sum$ Bd C_i are in general position. Such exists by Lemma 8. Each component of Bd $R \cdot \sum$ Bd C_i is a polygonal simple closed curve. It will be assumed that if R' is a polyhedral regular free-manifold in M containing J in its interior such that J can be shrunk to a point in R', and Bd R', $\sum$ Bd C_i are in general position, then Bd $R' \cdot \sum$ Bd C_i has no fewer components than does Bd $R \cdot \sum$ Bd C_i. This last set is assumed to be nonempty. The case where it is empty is covered in step 9.

3. There is no curve K in Bd $R\cdot\sum$ Bd C_i which bounds a disk D in one of the surfaces Bd C_j so that D misses J. For, if so, K could be chosen so that Bd $R\cdot$Int $D=\emptyset$. One then proceeds as in the proof of Lemma 7 to reduce the number of components of Bd $R\cdot\sum$ Bd C_i. That is, in case $R\cdot$Int $D=\emptyset$ one adds a cube to R; in case Int $D\subseteq$Int R and K separates some component of Bd R, one cuts R into 2 pieces, keeping the half that contains J and adjusting it slightly near D; and in case Int $D\subseteq$Int R but K separates no component of Bd R, one removes a handle from R. In any event, the resulting regular free-manifold would contradict the assumption in step 2.

4. Let j be any integer for which Bd $R\cdot$Bd $C_j\neq\emptyset$. There is a simple closed curve K_j in this intersection which bounds a disk D_j in Bd C_j for which Bd $R\cdot$Int $D_j=\emptyset$. Since J can be shrunk to a point in R, D_j must contain an even number of a_j, b_j, c_j, d_j, and by step 3 this number must be 2. Then, Int $D_j\subseteq$Int R. There is another simple closed curve K_j' in Bd $R\cdot$Bd C_j which bounds a disk D_j' in Bd C_j such that Int $D_j'\subseteq$Int R, Int $D_j\cdot$Int D_j' $=\emptyset$, and D_j' contains the other two of a_j, b_j, c_j, d_j. Note that, by step 3, each component of Bd $R\cdot$Bd C_j separates $J\cdot D_j$ from $J\cdot D_j'$ in Bd C_j.

5. Let j be as in step 4, and suppose that there is a polygonal arc α_j from a point x_j in Int D_j to a point y_j in Int D_j' such that Int $\alpha_j\subseteq$Int C_j, $\alpha_j\subseteq$Int R, and α_j is unknotted in C_j (i.e., α_j is contained in a polyhedral disk in C_j whose intersection with Bd C_j is its boundary). Then, there is a polyhedral regular free-manifold R' whose boundary is in general position with respect to $\sum$ Bd C_i, whose interior contains $J+C_j$ and is such that, if $i\neq j$, then Bd $C_i\cdot$Bd R' has the same number of components as does Bd $C_i\cdot$Bd R. The manifold R' is obtained by deforming R as explained in the next paragraph.

There is a polyhedral cube H in C_j such that Int $\alpha_j\subseteq$Int H, Bd $H\cdot$Bd C_j consists of 2 disjoint disks E and F, where $E\subseteq$Int D_j, Int $E\supseteq x_j+J\cdot D_j$ and $F\subseteq$Int D_j', Int $F\supseteq y_j+J\cdot D_j'$, and $H\subseteq$Int R. There is a piecewise linear homeomorphism h of M onto M which is fixed on $E+F+\alpha_j$ and outside a small neighborhood of C_j and which takes H onto C_j. The required R' is $h(R)$. By applying, if necessary, a finite sequence of such homeomorphisms to R, it may be assumed that R^* is a polyhedral regular free-manifold containing J in its interior such that Bd R^* and $\sum$ Bd C_i are in general position and if Bd $R^*\cdot$Bd $C_j\neq\emptyset$, then each component of this intersection separates c_j+d_j from a_j+b_j in Bd C_j. Note that it is not required that J lie trivially in R^*. The notation is chosen so that $c_j+d_j\subseteq D_j'$ and $a_j+b_j\subseteq D_j$.

6. It is shown in this step that if K is a component of Bd $R^*\cdot\sum$ Bd C_i, then K does not separate any component of Bd R^*. For, if so, K could be chosen to bound a disk D in Bd R^* such that Int $D\cdot\sum$ Bd $C_i=\emptyset$. Suppose $K\subseteq$Bd C_j.

The first possibility is that Int $D\subseteq$Int C_j-J. But D must separate the arc c_jd_j from the arc a_jb_j in C_j, since $K=$Bd D separates c_j+d_j from a_j+b_j

in Bd C_j. Then C_j is the sum of polyhedral cubes H_1 and H_2 such that $H_1 \cdot H_2 = D$, Int $a_j b_j \subseteq$ Int H_1, and Int $c_j d_j \subseteq$ Int H_2. Hence, $c_j d_j$ is homotopic in $C_j - a_j b_j$ to an arc in Bd C_j. This is in contradiction to Lemma 6 of [6].

Finally, it might happen that Int $D \subseteq M - (J + \sum C_i)$. Let D' be a disk in Bd C_j such that $K = \text{Bd } D'$ and $D' \cdot J = c_j + d_j$. Then, $S = D + D'$ is a polyhedral 2-sphere in M which is pierced by J at two points and otherwise misses J. The component of $J - (c_j + d_j)$ contained in C_j lies in one component of $M - S$, while the other component of $J - (c_j + d_j)$ must lie in the other component of $M - S$. But this is clearly not the case, since any arc in Int C_j from Int $c_j d_j$ to Int $a_j b_j$ will miss S. The assertion at the beginning of step 6 follows. In particular, no simply-connected component of Bd R^* can intersect $\sum$ Bd C_i.

7. It will be assumed from this point that no component of Bd R^* lies in any of the sets Int C_j. This is justified by filling the holes in each C_j. That is, suppose S is a component of Bd R^* such that $S \subseteq$ Int C_j. Let M^* be the closure of the component of $C_j - S$ not containing Bd C_j. Then M^* is a 3-cell or a cube with a tubular hole, $M^* \cdot R^* = \text{Bd } M^* \cdot \text{Bd } R^* = S$ and $M^* + R^*$ is a regular free-manifold.

This last assertion is clear if M^* is a 3-cell. If S is not a 2-sphere, let D be a polyhedral disk in M which forms a handle for R^*, whose boundary is in S, and such that Int D and Bd C_j are in general position. It suffices to show that $M^* + D$ can be embedded in E^3. One does this by using standard techniques to construct a polyhedral punctured cube about $M^* + D$ in M, beginning with C_j.

Here and in step 8, consider a fixed j such that Bd $R^* \cdot$ Bd $C_j \neq \emptyset$ and let C be a component of Bd R^* for which $C \cdot$ Bd $C_j \neq \emptyset$. Since no component of $C \cdot$ Bd C_j bounds a disk in C and since Bd C_j separates M, $C \cdot$ Bd C_j has at least two components and each component of $C_j \cdot$ Bd R^* is an annulus ring. Since each component of Bd $R^* \cdot$ Bd C_j can be shrunk to a point in $C_j - a_j b_j$ and in $C_j - c_j d_j$, the following holds: any simple closed curve in $C_j \cdot$ Bd R^* can be shrunk to a point in $C_j - a_j b_j$ and in $C_j - c_j d_j$.

8. If it can now be shown that the arc α_j described at the beginning of step 5 exists for this j, it will follow as at the end of step 5 that there is a polyhedral regular free-manifold whose interior contains J and whose boundary misses $\sum$ Bd C_i. Then, step 9 will apply and the proof will be complete. The existence of such an arc is shown now.

Let $c_j v d_j$ be a polygonal arc from c_j to d_j in Int D_j'. It would be convenient if this arc were in the position shown in Figure 1(a). The general situation is indicated in Figure 1(b), where $c_j v d_j$ spirals around Bd C_j several times. However, one may simplify the position of $c_j v d_j$ if he is willing to complicate the arcs $a_j b_j$ and $c_j d_j$ somewhat.

Consider now just the cube C_j of Figure 1(b). There is a homeomorphism h of C_j onto itself which wraps the left end of $c_j d_j$ around the left end of $a_j b_j$

as in Figure 1(c), leaves c_j, d_j, and a_jb_j fixed, and takes c_jvd_j into the position also indicated in Figure 1(c).

Consider this last figure. There is a polyhedral disk D^* whose interior is in Int C_j, whose boundary is $h(c_jd_j+c_jvd_j)$, and which is pierced by a_jb_j as in this figure. One such intersection point, indicated by p, is at the extreme bottom point of a_jb_j, and the rest are near Bd C_j. Note that $h^{-1}(\text{Bd } D^*) \subseteq \text{Int } R^*$, so it may be assumed that $h^{-1}(\text{Int } D^*)$ and Bd R^* are in general position. It will be shown that no component K of $\text{Int } D^* \cdot h(C_j \cdot \text{Bd } R^*)$ separates p from Bd D^* in D^*. From this, it will be immediate that the required arc can be found in $h^{-1}(D^*)+a_jb_j$. Suppose that one of these curves K bounds a disk D in Int D^* that contains p. According to step 7, K can be shrunk to a point in $C_j-a_jb_j$. The next paragraph completes step 8 by showing that this is impossible.

It is convenient to consider C_j in E^3 and to note that a loop in $C_j-a_jb_j$ can be shrunk to a point in $C_j-a_jb_j$ if and only if it can be shrunk to a point in E^3-J^* (see Figure 1(a)). The fundamental group G of E^3-J^* has generators x and z with the single relation $xzx=zxz$. These generators are indicated in Figure 1(a). If K were trivial in E^3-J^*, then for some integer n the element x^nz would be the identity of G. That this is not the case is shown by the representation ϕ of G defined by:

$$\phi(x) = (12)(3),$$

$$\phi(z) = (13)(2).$$

9. If $\text{Bd } R^* \cdot \sum \text{Bd } C_i = \varnothing$, then, by Lemmas 2 and 3,

$$[R^* - \sum (R^* \cdot \text{Int } C_i)] + \sum C_i$$

is a 3-sphere or a polyhedral regular free-manifold whose interior contains T_1, and the proof of Theorem 1 is completed by Lemma 7.

As a special case of Theorem 1, the following holds:

THEOREM 2. *Let M be a compact, connected absolute 3-manifold such that each polyhedral simple closed curve in M lies trivially in a solid torus* (*of genus* 1) *in M. Then M is topologically S^3.*

6. **Extension of results to homologically trivial 3-manifolds.** Theorem 1 was proved under the assumption that the 3-manifold M was simply-connected in a strong sense. Actually, more was assumed than was necessary in the proof of this theorem. This permitted a few simplifications in the proof, however. If M is a compact, connected, 3-manifold for which $H_1(M; Z)=0$, then by Poincaré duality, $H_2(M; Z)=0$. It follows that compact, polyhedral 2-manifolds separate M.

The result analogous to Lemma 6 is:

LEMMA 6′. *Let R_1 be a regular free-manifold and R_2 a punctured cube, where*

$R_1 \cdot R_2$ is an annulus ring A in Bd $R_1 \cdot$Bd R_2. *If R_1+R_2 can be embedded in a 3-manifold M for which $H_1(M; Z)=0$, then R_1+R_2 is a regular free-manifold.*

Note that in the proof of Theorem 1, one needs only Lemma 6′, the assumption that J circles each handle of R an even number of times (see step 4 of Theorem 1), and the fact that compact, polyhedral 2-manifolds separate M. Hence, the following holds:

THEOREM 3.(2) *Let M be a compact, connected, absolute 3-manifold for which $H_1(M; Z)=0$ and such that each polyhedral simple closed curve lies in a regular free-manifold in M in a homologically trivial manner. Then, M is homeomorphic to S^3.*

The interested reader should have no trouble in supplying the details omitted here.

EXAMPLE. It is of interest to note that the condition that $H_1(M; Z)$ be trivial is essential in Theorem 3. That is, there is a compact, connected 3-manifold M such that each polyhedral simple closed curve lies in the interior of a solid torus of genus 1 in M in a homologically trivial manner, yet $H_1(M; Z)$ does not vanish. The following M is a lens space, and is one of many examples with the above properties that could be given.

Let T_i $(i=1, 2)$ be a solid torus of genus 1. Let M be the 3-manifold obtained by identifying Bd T_2 with Bd T_1 in such a way that the meridianal simple closed curve J_1 on Bd T_2 circles T_1 three times longitudinally and once meridianally, and the longitudinal simple closed curve J_2 on Bd T_2 circles T_1 twice longitudinally and once meridianally. Note that $H_1(M; Z)$ is Z_3, the cyclic group of order three. Consider M as having a fixed triangulation.

It will be shown that if K is a simple closed curve in M, then there is a homeomorphism of M onto itself which throws K onto a simple closed curve K_3 in Int T_1 which circles T_1 an even number of times. This homeomorphism will be the composition of three homeomorphisms of M onto M. The first of these is fixed outside a small neighborhood of T_1, and pushes K onto a simple closed curve K_1 in Int T_2.

Now let N be a small polyhedral tubular neighborhood in M of the longitudinal curve J_2 (N is topologically a solid torus of genus 1). The second homeomorphism is the identity on T_1 and takes K_1 onto K_2 in Int N.

Lastly, there is a homeomorphism of M which is fixed outside a small neighborhood of T_2 and throws N onto a torus in Int T_1 which circles T_1 twice longitudinally. This homeomorphism takes K_2 onto a simple closed curve K_3 in Int T_1 which circles T_1 an even number of times. The property of M asserted above follows.

7. **An example.** Let M be a compact, connected 3-manifold with boundary. If M can be embedded in a simply-connected 3-manifold, then M is

(2) *Added in proof.* J. J. Andrews has pointed out that, except in Lemma 8, one needs only that the simple closed curve circles no handle of the regular free-manifold exactly once.

simply-connected mod its boundary. If each component of Bd M is a 2-sphere and M is simply-connected mod Bd M, then M can be embedded in a compact, connected simply-connected 3-manifold. Lemma 6 concerns a situation in which a 3-manifold with boundary which is simply-connected mod its boundary can be embedded in E^3. An example M_1 is given here to show that if one component of Bd M_1 fails to be simply-connected, then M_1 may not be embeddable in any simply-connected 3-manifold, even though M_1 is simply-connected mod Bd M_1 and its boundary is connected and of genus 1. Further, there is a compact, connected 3-manifold M_2 whose boundary is of genus 1, such that M_2 can be embedded in E^3 and the fundamental group of M_2 is isomorphic to the fundamental group of M_1. The manifold M_2 is a cube with a knotted tubular hole. In [1], Alexander has given examples of topologically different compact, connected absolute 3-manifolds with the same fundamental group.

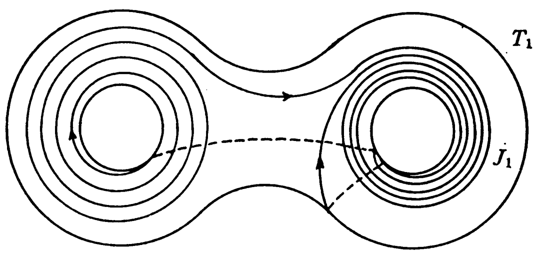

FIG. 2(a)

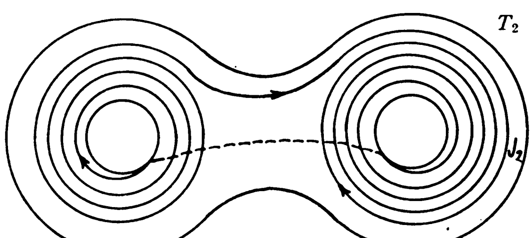

FIG. 2(b)

Let T_i ($i=1, 2$) be a double torus (cube with 2 handles). Each of M_1 and M_2 will be obtained by attaching a cube C_i to T_i so that $T_i \cdot C_i = \text{Bd } T_i \cdot \text{Bd } C_i$ is an annulus ring A_i. Let J_i be one component of Bd A_i. The simple closed curve J_i circles T_i as shown in Figures 2(a) and 2(b). The fundamental group of M_i is generated by elements a and b subject to the relation $a^4b^5=1$, since the fundamental group of T_i is the free group on 2 generators a and b.

First, consider Figure 2(b). One can attach a cube C' to M_2 along an annulus ring in such a way that M_2+C' is a simply-connected manifold bounded by a 2-sphere, $T_2 \cdot C' = \text{Bd } T_2 \cdot \text{Bd } C'$ is an annulus ring, and $T_2 + C'$ is a solid torus. By Lemma 6, $T_2 + C' + C_2$ is a cube, and $T_2 + C_2 = M_2$ can be embedded in E^3.

Now, suppose that $M_1 = T_1 + C_1$ is contained in a simply-connected 3-manifold M as a polyhedral subset. It follows from Theorem 1 of [13] and Dehn's Lemma of [12] that there is a polygonal simple closed curve J in Bd M_1 which does not separate Bd M_1 but which bounds a polyhedral disk D in M such that $\text{Bd } M_1 \cdot \text{Int } D = \emptyset$. In case M is S^3, this is shown in [8]. Now, either $\text{Int } D \subseteq \text{Int } M_1$ or $M_1 \cdot \text{Int } D = \emptyset$. If the former were the case, then M_1 would be obtained by adding a handle to a simply-connected 3-manifold bounded by a 2-sphere, and the fundamental group of M_1 would be infinite cyclic. It is easy, however, to construct examples of nonabelian groups generated by 2 elements a and b satisfying $a^4b^5 = 1$. Hence, $M_1 \cdot \text{Int } D = \emptyset$.

If D is thickened slightly to obtain a polyhedral cube C such that $\text{Int } D \subseteq \text{Int } C$ and $C \cdot M_1 = \text{Bd } C \cdot \text{Bd } M_1 = A$, an annulus ring, then $M_1 + C$ is a compact, connected, simply-connected 3-manifold bounded by a 2-sphere. Hence, the assumption that M_1 can be piecewise linearly embedded in M will be shown to be incorrect if it can be shown that it is impossible to attach a cube to M_1 in this manner so that the resulting manifold is simply-connected.

If this were possible, then there would be a polygonal simple closed curve J in $\text{Bd } T_1 - A_1$ such that $\text{Bd } T_1 - (J + J_1)$ is connected and, if the element of the fundamental group of T_1 corresponding to J is $W(a, b)$, then the only group with generators a and b satisfying $a^4b^5 = 1 = W(a, b)$ is the trivial group. It will be shown that there is no such J.

There are polygonal simple closed curves B and C in $\text{Bd } T_1 - A_1$ such that B circles the left handle of T_1 once, the right handle of T_1 3 times, and C circles the left handle of T_1 4 times, and the right handle not at all. Further, B and C meet at only 1 point, where they cross. Then, B and C generate the fundamental group of Bd M_1, where this last set is a 2-dimensional torus. Recalling that the number of times that a simple closed curve in Bd M_1 circles longitudinally is relatively prime to the number of times it circles meridianally, one finds that $W(a, b)$ can, by use of the relation $a^4b^5 = 1$, be put into the form $(ab^3)^i a^{4j}$, where i and j are relatively prime integers. Hence, if the simple closed curve J described in the previous paragraph were to exist, there would have to be relatively prime integers i and j such that if the relations $a^4b^5 = 1 = (ab^3)^i a^{4j}$ are added to the free group on generators a and b, then the resulting group is trivial.

If the relation that a and b commute is added to the above relations, an abelian group on generators a and b is obtained, where $a^4b^5 = 1 = a^{4j+i}b^{3i}$. This group is trivial if and only if

$$\begin{vmatrix} 4 & 5 \\ 4j + i & 3i \end{vmatrix} = 7i - 20j = \pm 1.$$

Hence, i is of the form $3+20x$ and j is of the form $1+7x$, where x is an integer. It is not necessary to consider separately the case $i=-3+20x$ and $j=-1+7x$.

In [7], examples are given to show that if k and m are relatively prime integers, and $k, m, n\neq 1$, then there exist nontrivial groups on elements u and v, where $u^k=v^m=(uv)^n=1$. Let G_x be such a group for $k=4$, $m=5$, and $n=3+20x$, so that $u^4=v^5=(uv)^{3+20x}=1$. The group G_x is also generated by the elements $a=u$ and $b=v^2$, where $a^4=b^5=(ab^3)^{3+20x}=1$. Thus, for each integer x there is a nontrivial group G_x generated by elements a and b such that $a^4b^5=1=(ab^3)^{3+20x}a^{4(1+7x)}$. This completes the proof that there is no piecewise linear homeomorphism taking M_1 into a simply-connected 3-manifold M. According to Theorem 10 of [3], there can be no homeomorphism taking M_1 into M.

The 3-manifold M_1 is simply-connected mod Bd M_1, since if G is a group generated by elements x and y such that $x^4y^5=xy^3=x^4=1$, then $G=\{1\}$.

QUESTION. The preceding example shows how little can be said of a compact, connected, orientable 3-manifold with boundary M from a knowledge of its fundamental group. However, if this group is a free group and if every compact, polyhedral 2-manifold separates M, then the structure of M is determined mod the Poincaré Conjecture. Perhaps the following can be proved without the Poincaré Conjecture: Let M_1 be a cube with handles and M_2 a cube, where $M_1\cdot M_2=\text{Bd } M_1\cdot\text{Bd } M_2$ is an annulus ring. Suppose that the 3-manifold with boundary $M'=M_1+M_2$ has a free fundamental group, and that every compact, polyhedral 2-manifold separates M'. Then, M' can be embedded in E^3.

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UNIVERSITY OF WISCONSIN,
MADISON, WISCONSIN
LOUISIANA STATE UNIVERSITY,
BATON ROUGE, LOUISIANA